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Correct Sampling of Diffraction Limited Images

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An image of the sky, when focused by a telescope, should be sampled with enough pixels per beam to Nyquist sample the highest spatial frequency in that image. Otherwise the resulting sampled image will be corrupted by aliasing artifacts. The point spread function (PSF) of the telescope optics acts like a low pass filter in the spatial frequency domain, so to determine the number of pixels needed per beam, one must first determine the maximum frequency passed by the point spread function.

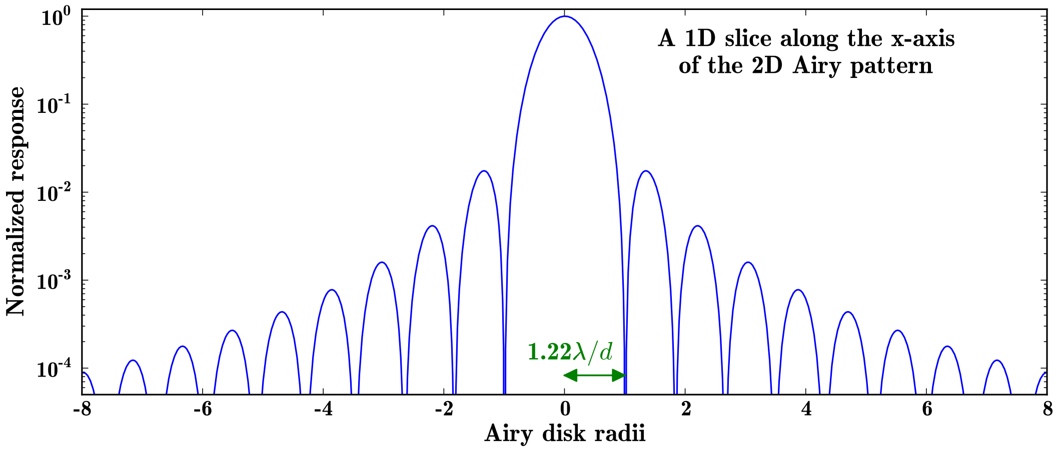
The point spread function of a uniformly illuminated aperture of diameter, d, is the Airy diffraction pattern:

$$p(\theta) = 2\left(\frac{J_1(2\pi d/\lambda \sin\theta)}{2\pi d/\lambda \sin\theta}\right)^2 \tag{1}$$

For telescopes with a field of view of less than a few degrees, the $\sin\theta$ terms in this equation can be approximated by θ, which is the angle on the sky from the center of the optic axis. The PSF given by the above equation, and the corresponding frequency response, given by taking a radial slice through the Fourier transform of the 2-dimensional Airy pattern, is shown in figure 1.

The angular separation between the first zero of the PSF and the peak of the PSF is $1.22\lambda/d$ radians, where λ is the wavelength of the observation, and d is the diameter of the telescope. This is henceforth referred to as the beam width. To make the diagram independent of the telescope size, the spatial coordinates of the PSF are shown as multiples of the beam width. Similarly, the spatial frequency coordinates of the frequency response are shown in units of cycles per beam width.

The frequency response plot reveals that the highest spatial frequency in the PSF is 1.22 cycles per beam. This frequency of 1.22 cycles per $1.22\lambda/d$ radians, is equivalent to 1.0 cycles per λ/d radians. This is expected, because d/λ it is the spatial frequency seen by a single baseline interferometer of length equal to the diameter of the telescope aperture.



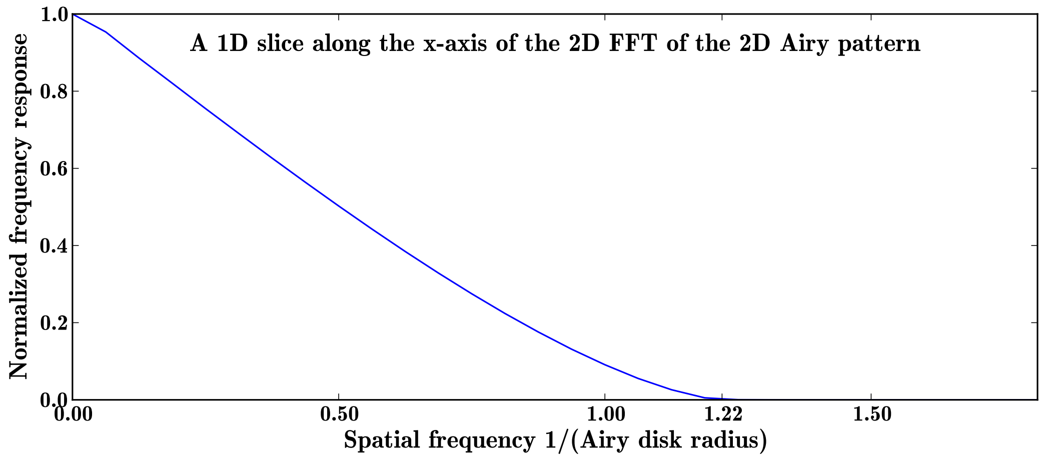


Figure 1: 1D radial slices through the 2D Airy PSF and its Fourier transform

In accordance with Nyquist/Shannon sampling theorem, to retain all of the information in an image at the focal surface of a telescope, it is necessary to sample the image with at least twice as many samples per beam as the maximum frequency in the PSF, expressed in cycles per beam. Rounding 2×1.22 cycles per beam to the next highest integer, shows that at least 3 samples per beam are needed.

If fewer than 3 samples are used per beam, then any features on the sky that have spatial frequencies between the Nyquist folding frequency of the chosen number of samples per beam, and 1.22 cycles per beam, are aliased to lower spatial frequencies, resulting in aliasing artifacts. This is illustrated in figure 2.

The top left image in this plot shows a simulated image of the sky, convolved with an airy pattern PSF. The simulated image is composed of 3 point sources that are too close together to be resolved apart by the PSF, and an isolated point source that illustrates the airy pattern PSF. The image is displayed in a log scale to reveal the otherwise dim side lobes of the PSF. The discrete Fourier transform of this image is shown in the image on its right. This shows

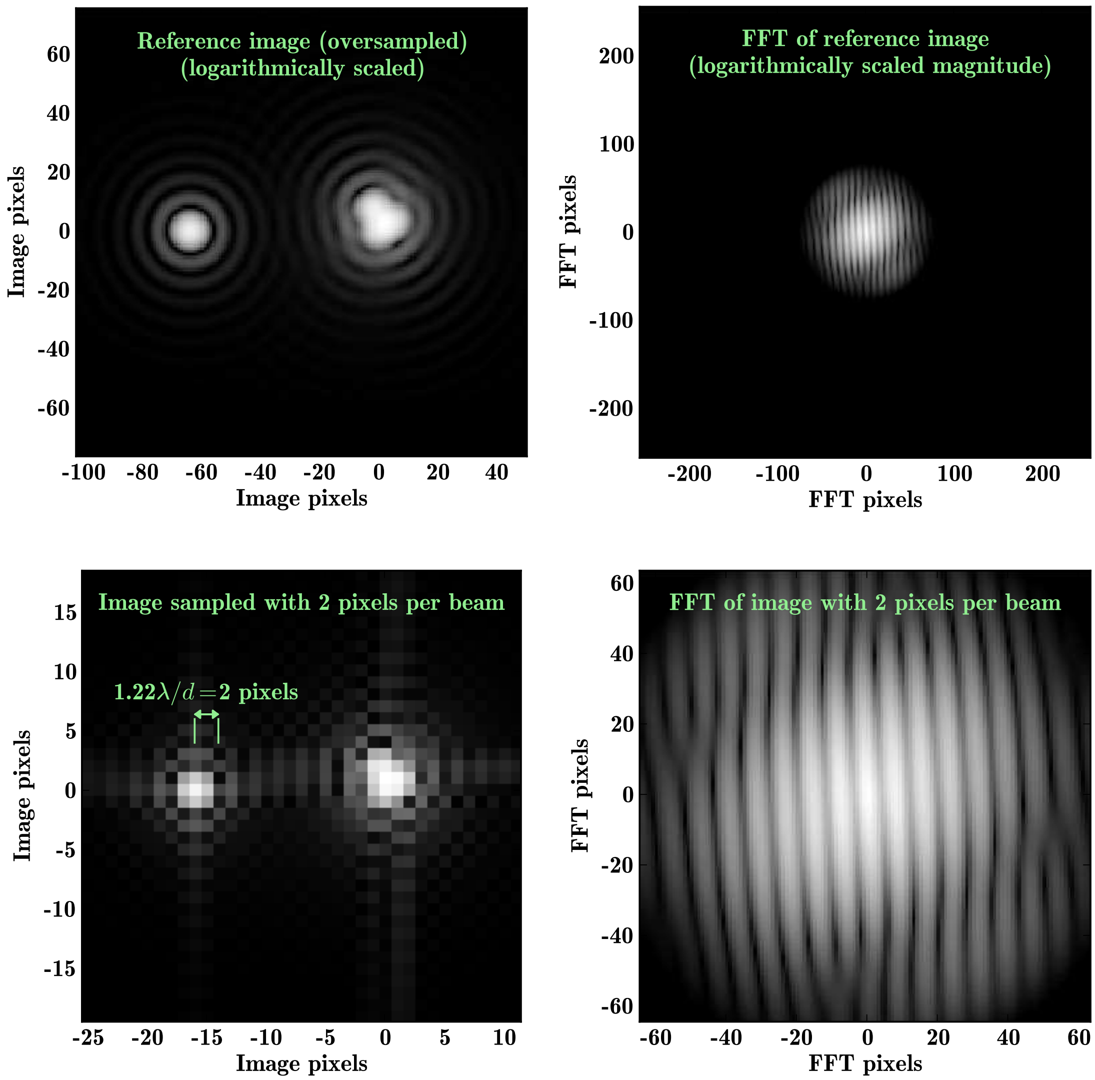


Figure 2: A simulated image and how it looks when sampled with 2 pixels per beam

the circularly symmetric airy pattern frequency response, bounded at a radius of 1.22 cycles per beam from the center, modulated by sine waves that correspond to the offset positions of the point sources. Again this has been shown in a log scale, this time to make the dim edges of the frequency response visible. It can be seen in the Fourier transform image that the whole of the PSF is sampled by the image. So there is no aliasing.

The two images in the bottom half of the plot are the same as those above, except that the simulated image is now sampled with just 2 pixels per beam. In the Fourier transform of this image, it is clear that the circular PSF frequency response has been truncated everywhere except at the corners of the image. As a result, the image plane image shows vertical and horizontal aliasing artifacts, in the form of periodically modulated lines of pixels radiating from each point source. Although these aliased artifacts are at a low level, they would be significant if they were from a bright point source, and they impinged on a nearby dim source. They could also result in mis-calibration, because the aliased artifacts contain some of the flux of the source, and this would be missed if flux was only integrated from close to the source, to avoid flux from nearby confusing sources.

When the same image is instead sampled with 3 pixels per beam, the result is as shown in figure 3. In this case the entire circular PSF frequency response is within the bounds of the Fourier transform, so the resulting image is completely free of aliasing artifacts.

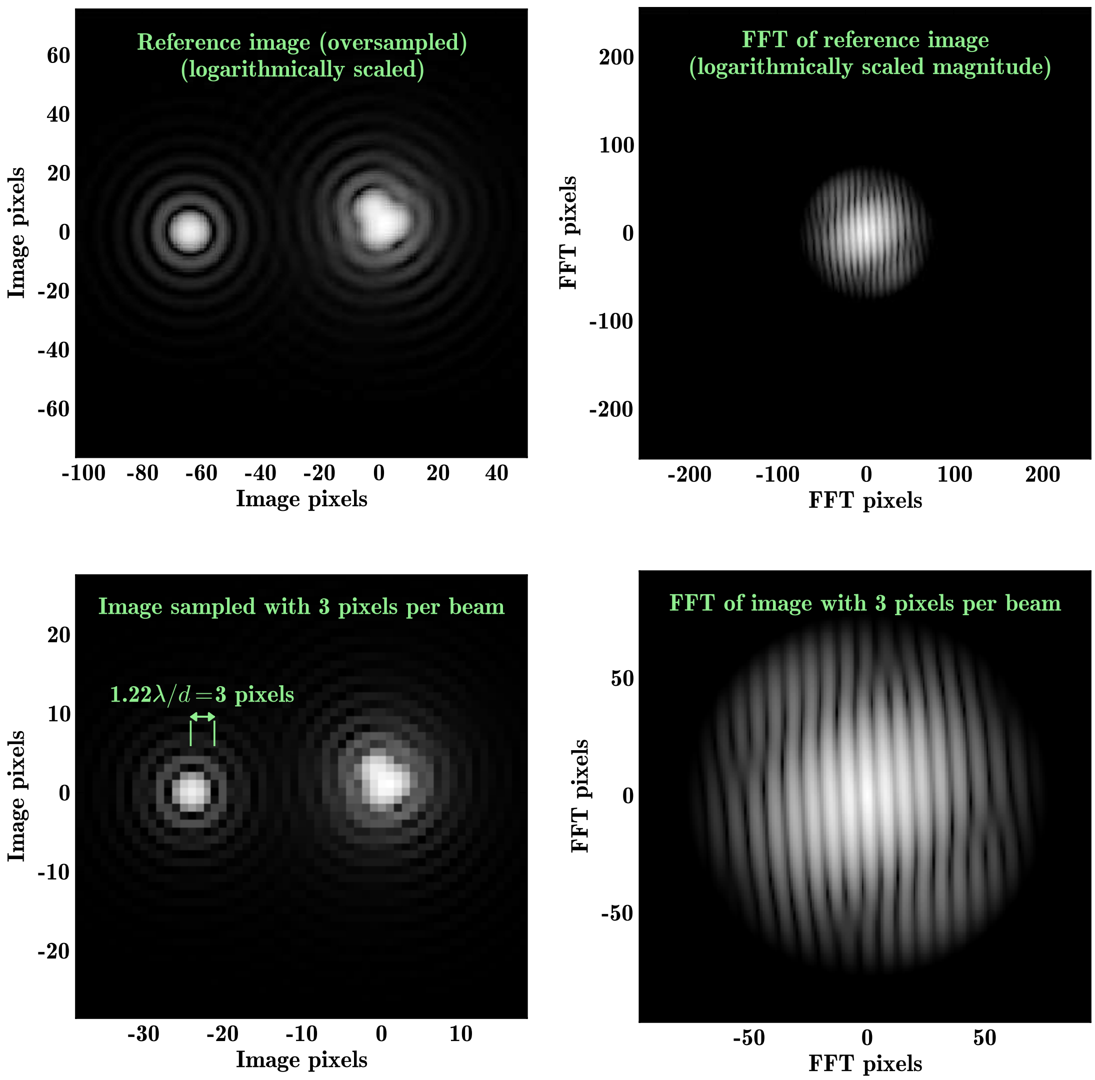


Figure 3: A simulated image and how it looks when sampled with 3 pixels per beam

In summary, a staring camera or a scanning instrument at the focus of a diffraction limited telescope should sample the sky with at least 3 samples per beam, where one beam is defined to be the radius of the airy disk, which is $1.22\lambda/d$ for a telescope with a circular aperture of diameter d, while observing the sky at a minimum wavelength of λ.